



Quantum electrodynamics and the origins of the exchange, dipole-dipole, and Dzyaloshinsky-Moriya interactions in itinerant fermion systems

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It is shown how the exchange interaction, the dipole-dipole interaction, and the Dzyaloshinsky-Moriya interaction between electronic spin-density fluctuations emerge naturally from a field-theoretic framework that couples electrons to the fluctuating electromagnetic potential. Semiquantitative estimates are given to determine when the dipole-dipole interaction, which is often neglected, needs to be considered and various applications are discussed, with an emphasis on weak ferromagnets and on helimagnets.

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I. INTRODUCTION

Understanding the origin of ferromagnetism was one of the success stories of applying quantum mechanics to solid-state systems. Classically, magnetic moments interact via the dipole-dipole interaction, which is much too weak to explain magnetic order at as high a temperature as is observed in, e.g., iron or nickel.¹ The explanation of this conundrum was found to be the exchange interaction mechanism, which leads to a spin-spin interaction that is governed by the Coulomb interaction via the Pauli principle. This was first understood in the context of atomic and molecular physics in 1920s and applied to solid-state physics in 1950s.¹ Somewhat ironically, a straightforward application of the exchange interaction concept leads to a spin-spin interaction that is too strong, as the relevant energy scale is the atomic scale, or roughly 100 000 K. Many-body and band-structure effects renormalize this scale and bring it down to the observed ferromagnetic scale of roughly 1000 K or lower.² This is still much larger than the dipole-dipole scale and the latter is often neglected in the discussion of ferromagnets. When it is considered, e.g., for its influence on the critical behavior,^{3–5} it is usually added phenomenologically to models that describe the exchange interaction. Another spin-spin interaction that has been of interest lately is the Dzyaloshinsky-Moriya (DM) interaction.^{6,7} It results (in systems with suitable lattice structures) from the spin-orbit interaction, has been derived from microscopic models, and is believed to be responsible for the helical magnetic order observed in MnSi and FeGe.⁸ Rough estimates show that the DM interaction and the dipole-dipole interaction are of about the same strength and should thus be considered together.⁹ Furthermore, in weak ferromagnets, which order only at low temperatures, all three interactions can be comparable in strength, which can make the dipole-dipole and DM interactions crucial.

In this paper we provide a comprehensive derivation of all of these effects within one unified framework, namely, a field-theoretic description of electrons and photons. Starting with finite-temperature quantum electrodynamics coupled to a field-theoretic description of finite-density quasirelativistic electrons we show that the exchange, dipole-dipole, and DM

interactions all appear naturally upon integrating out the photons. The exchange and DM interactions arise from integrating out the scalar part of the electromagnetic potential; the dipole-dipole interaction from integrating out the vector potential. Furthermore, the DM and dipole-dipole interactions are indeed of the same order in the relativistic corrections to the Schrödinger equation (i.e., of second order in v_F/c , with v_F the Fermi velocity and c the speed of light or of second order in the fine-structure constant α).

Integrating out the fermions then leads to an effective theory for quantum magnets that generalizes and replaces the Hertz-Millis theory^{10,11} and its generalizations.¹² More generally, the theory provides a derivation of spin-spin interactions in itinerant Fermi systems in general, whether or not they are in a parameter regime where they develop long-range magnetic order. Our results are therefore relevant, for instance, for fermionic atoms in optical traps or on optical lattices.¹³

This paper is organized as follows. In Sec. II we consider, as a warm up and to introduce various concepts, classical magnets, and show how the vector-potential coupling to the magnetization gives rise to the dipole-dipole interaction. In Sec. III we develop the technical machinery for dealing with quantum magnets and provide the derivations mentioned above. In Sec. IV we discuss our results and provide a summary and conclusion. Some technical details are relegated to various appendices.

II. EFFECTIVE ACTION FOR CLASSICAL FERROMAGNETS AND HELIMAGNETS

We now proceed to derive an effective action for magnets that includes the effects of the fluctuating electromagnetic potential. We first consider the classical case as a warm up; we will generalize to the quantum case in Sec. III.

A. Dipole-dipole interaction

Consider a classical model for a ferromagnet with a three-component order parameter \mathbf{M} . In addition to the field $\mathbf{M}(\mathbf{x})$

we need to consider the electromagnetic vector potential $\mathbf{A}(\mathbf{x})$ and the partition function Z is given by

$$Z = \int D[\mathbf{M}, \mathbf{A}] e^{S[\mathbf{M}, \mathbf{A}]} \quad (2.1a)$$

$$\equiv \int D[\mathbf{M}] e^{-\mathcal{F}[\mathbf{M}]/T}. \quad (2.1b)$$

The model is defined by specifying the action S and in Eq. (2.1b) we have anticipated integrating out the vector potential to obtain an effective action \mathcal{F} in terms of the order parameter only. T denotes the temperature, so \mathcal{F} is the free energy in mean-field approximation. Throughout this paper we will use units such that Boltzmann's constant and Planck's constant are equal to unity, $k_B = \hbar = 1$.

For the order-parameter part of S , we consider an $O(3)$ -symmetric ϕ^4 theory,

$$S_M = \frac{-1}{T} \int_V d\mathbf{x} \left\{ \frac{t}{2} \mathbf{M}^2(\mathbf{x}) + \frac{a}{2} [\nabla \mathbf{M}(\mathbf{x})]^2 + \frac{u}{4} \mathbf{M}^4(\mathbf{x}) \right\}. \quad (2.2a)$$

S_M represents a Landau-Ginzburg-Wilson (LGW) theory of an isotropic ferromagnet with volume $V \rightarrow \infty$. The parameter t contains the exchange interaction that leads to a magnetic ordering transition. In mean-field approximation this transition occurs at $t=0$, with $t > 0$ describing the paramagnetic phase, and $t < 0$ the ferromagnetically ordered one. $a > 0$ and $u > 0$ are two additional model parameters and $(\nabla \mathbf{M})^2 = \partial_i M_j \partial^i M^j$. Here, and throughout the paper, summation over repeated vector, tensor, and spinor indices is implied unless otherwise noted. Note that S_M is separately invariant under rotations in \mathbf{M} (spin) space and real space, respectively.

The magnetization¹⁴ \mathbf{M} couples linearly to the curl of the magnetic vector potential \mathbf{A} ,

$$S_c = \frac{\mu_B}{T} \int_V d\mathbf{x} \mathbf{M}(\mathbf{x}) \cdot [\nabla \times \mathbf{A}(\mathbf{x})] \quad (2.2b)$$

with $\mu_B = e/2m_e c$ the Bohr magneton in terms of the electron charge e , the electron mass m_e , and the speed of light c . \mathbf{A} and $\nabla \times \mathbf{A}$ transform as vectors in real space, and therefore S_c is invariant only under corotations of spin space and real space. It is this coupling of the magnetization to the fluctuating vector potential that allows one to consider the magnetization as having a particular direction in real space. The vector potential is governed by

$$S_A = \frac{-1}{8\pi T} \int_V d\mathbf{x} \left\{ [\nabla \times \mathbf{A}(\mathbf{x})]^2 + \frac{1}{\rho} [\nabla \cdot \mathbf{A}(\mathbf{x})]^2 \right\} \quad (2.2c)$$

with ρ any real number. The first term in Eq. (2.2c) is the magnetic energy and the second term with coupling constant $1/\rho$ is a gauge fixing term. One popular choice is $\rho=0$, which enforces a Coulomb gauge, $\nabla \cdot \mathbf{A}=0$; another one is the Feynman gauge, $\rho=1$.^{15,16} Either choice ensures a finite \mathbf{A} propagator. In Coulomb gauge, it is

$$\langle A_i(\mathbf{k}) A_j(-\mathbf{k}) \rangle = 4\pi T \frac{\delta_{ij} - \hat{k}_i \hat{k}_j}{k^2}. \quad (2.3)$$

The vector potential can now be integrated out exactly, which leads to an effective action in terms of \mathbf{M} only. Alternatively, we can consider the magnetic induction $\mathbf{B} = \nabla \times \mathbf{A}$ the fundamental field to be integrated out. In that case, the gauge fixing condition needs to be replaced by a constraint that enforces the Maxwell equation $\nabla \cdot \mathbf{B}=0$. That is, Eqs. (2.2b), (2.2c), and (2.3) are replaced by¹⁷

$$S_c = \frac{\mu_B}{T} \int_V d\mathbf{x} \mathbf{M}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}), \quad (2.2b')$$

$$S_A = \frac{-1}{8\pi T} \int_V d\mathbf{x} \left\{ \mathbf{B}^2(\mathbf{x}) + \frac{1}{\rho} [\nabla \cdot \mathbf{B}(\mathbf{x})]^2 \right\}_{\rho \rightarrow 0}, \quad (2.2c')$$

$$\langle B_i(\mathbf{k}) B_j(-\mathbf{k}) \rangle = 4\pi T (\delta_{ij} - \hat{k}_i \hat{k}_j). \quad (2.3')$$

Either way we find

$$\begin{aligned} \mathcal{F} = & \int_V d\mathbf{x} \left\{ \frac{1}{2} (t - 4\pi\mu_B^2) \mathbf{M}^2(\mathbf{x}) + \frac{a}{2} [\nabla \mathbf{M}(\mathbf{x})]^2 + \frac{u}{4} \mathbf{M}^4(\mathbf{x}) \right\} \\ & + 2\pi\mu_B^2 \sum_k d_{ij}(\mathbf{k}) M_i(\mathbf{k}) M_j(-\mathbf{k}) \end{aligned} \quad (2.4a)$$

with

$$d_{ij}(\mathbf{k}) = \hat{k}_i \hat{k}_j. \quad (2.4b)$$

The terms generated by integrating out the vector potential we recognize as the leading contribution to the dipole-dipole interaction^{3,5} plus a shift of the Landau parameter t by $4\pi\mu_B^2$. The scalar potential $\varphi(\mathbf{x})$, whose gradient is the electric field, does not lead to any magnetic interactions in a classical theory. This changes once the system is treated quantum mechanically, see Sec. III below.

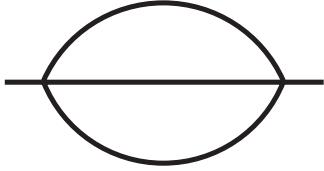
B. Renormalization and higher order terms

The dipole-dipole operator d_{ij} , Eq. (2.4b), transforms as a rank-2 tensor in momentum (or real) space and \mathbf{M} transforms as a vector in spin space. Consequently, the dipole-dipole interaction is invariant under corotations in real space and spin space. This raises the question of other terms in the action that have the same symmetry properties. For instance, $(\nabla \cdot \mathbf{M})^2$ is allowed by symmetry. This term and terms of higher order in the gradient are generated by a renormalization of the action \mathcal{F} , as we now proceed to show.

A renormalization of the action \mathcal{F} generates additional terms by virtue of the anisotropic \mathbf{M} propagator, which now reads

$$\langle M_i(\mathbf{k}) M_j(-\mathbf{k}) \rangle = \frac{\delta_{ij} - \hat{k}_i \hat{k}_j}{t - 4\pi\mu_B^2 + ak^2} + \frac{\hat{k}_i \hat{k}_j}{t + ak^2}. \quad (2.5)$$

For instance, two-loop diagrams of the structure shown in Fig. 1 both renormalize $d_{ij}(\mathbf{k})$ and lead to a new vertex

FIG. 1. A diagram that generates a $(\nabla \cdot \mathbf{M})^2$ term.

$$\sum_{\mathbf{k}} k_i k_j M_i(\mathbf{k}) M_j(-\mathbf{k}), \quad (2.6)$$

as well as to higher order anisotropic gradient terms. Equation (2.6) represents the $(\nabla \cdot \mathbf{M})^2$ term that was mentioned above. Conversely, if one starts with a theory that contains a $(\nabla \cdot \mathbf{M})^2$ term, which is allowed by symmetry and hence should be included in any Landau theory, then the leading dipole-dipole term will be generated in perturbation theory even if it was not included in the bare action. The complete LGW action for a classical, isotropic Heisenberg ferromagnet, up to terms quadratic in gradients and quartic in the order parameter, thus reads

$$\begin{aligned} \mathcal{F} = & \int_V d\mathbf{x} \left\{ \frac{r}{2} \mathbf{M}^2(\mathbf{x}) + \frac{a}{2} [\nabla \mathbf{M}(\mathbf{x})]^2 + \frac{u}{4} \mathbf{M}^4(\mathbf{x}) \right\} \\ & - \frac{d_0}{2} \int_V d\mathbf{x} d\mathbf{y} M_i(\mathbf{x}) d_{ij}(\mathbf{x} - \mathbf{y}) M_j(\mathbf{y}) + \frac{d_2}{2} \int_V d\mathbf{x} [\nabla \cdot \mathbf{M}(\mathbf{x})]^2. \end{aligned} \quad (2.7a)$$

Here $d_{ij}(\mathbf{x} - \mathbf{y})$ is the Fourier transform of $d_{ij}(\mathbf{k})$ in Eq. (2.4b), namely,

$$d_{ij}(\mathbf{x} - \mathbf{y}) = \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|\mathbf{x} - \mathbf{y}|}. \quad (2.7b)$$

r is the bare distance from the critical point in the effective LGW theory that takes into account the effect of the vector potential, and a , u , d_0 , and d_2 are the remaining Landau parameters. As is clear from the above discussion, one expects the bare values of d_0 and d_2 to be small of order $1/c^2$ compared to the other parameters in natural units. These two parameters are usually set equal to zero in elementary treatments of classical Heisenberg ferromagnets.

C. Dzyaloshinsky-Moriya interaction and helimagnets

The terms in the action so far are all even in the gradient operator and hence invariant under spatial inversion. The spin-orbit interaction can eliminate this requirement by coupling the electron spins to the underlying lattice, provided the crystal structure is not inversion invariant. Dzyaloshinsky⁷ and Moriya⁶ showed that to linear order in the spin-orbit interaction g_{so} the relevant term is

$$\mathcal{F}_{\text{DM}} = \frac{-c_{\text{DM}}}{2T} \int d\mathbf{x} \mathbf{M}(\mathbf{x}) \cdot [\nabla \times \mathbf{M}(\mathbf{x})] \quad (2.8)$$

with $c_{\text{DM}} \propto g_{\text{so}}$. At a classical level, this term is purely phenomenological. DM showed how to derive it in the context of quantum mechanics and in Sec. III we will see how it arises in the context of field theory.

III. QUANTUM SYSTEMS

We now turn to a quantum-mechanical description of itinerant fermion systems in general and certain types of magnetism, in particular. We will show how the exchange interaction, the dipole-dipole interaction, and the DM interaction naturally arise in the context of a field-theoretic description of itinerant electrons. The former two lead to ferromagnetism and the latter, if it is present, to helimagnetism.

A. Action

In Appendix A we list the complete action for free quasirelativistic electrons, to order $1/c^2$, coupled to the electromagnetic field. Several terms in the complete action are not relevant for our present purposes. The only effect of the Darwin term, Eq. (A4g), is to modify the Coulomb interaction on length scales given by the electronic Compton wavelength $\lambda_e = 1/m_e c$. The relativistic mass enhancement, the last term in Eq. (A4b), is a higher order gradient term that is small compared to terms of the same form that are generated by renormalizing the final effective action. Finally, the Landau diamagnetic terms, Eq. (A4f), give rise to diamagnetism and, in the presence of an external magnetic field, Landau levels. These effects are physically very different from ferromagnetism or helimagnetism and we do not consider them here. Finally, in quantum electrodynamics the Fadeev-Popov ghost field does not couple to any other fields. Its only effect is to subtract the contribution of the unphysical longitudinal photon polarization to the free energy and one has to keep it only if one is interested in the absolute value of the latter. Neglecting all of these terms, we thus consider the following action:¹⁸

$$\begin{aligned} S(\bar{\psi}, \psi; A_\mu) = & \int d\mathbf{x} \bar{\psi}_\sigma(\mathbf{x}) \left(-\partial_\tau + \frac{1}{2m_e} \nabla^2 + \mu \right) \psi_\sigma(\mathbf{x}) \\ & + \frac{1}{8\pi} \int d\mathbf{x} A_\mu(\mathbf{x}) \left(\frac{1}{c^2} \partial_\tau^2 + \nabla^2 \right) A_\mu(\mathbf{x}) \\ & - ie \int d\mathbf{x} \varphi(\mathbf{x}) n(\mathbf{x}) + \mu_B \int d\mathbf{x} \mathbf{B}(\mathbf{x}) \cdot \mathbf{n}_s(\mathbf{x}) \\ & + \frac{-e}{4m_e^2 c^2} \int d\mathbf{x} \bar{\psi}_{\sigma_1}(\mathbf{x}) \boldsymbol{\sigma}_{\sigma_1 \sigma_2} \cdot [\nabla \varphi(\mathbf{x}) \times \nabla] \psi_{\sigma_2}(\mathbf{x}). \end{aligned} \quad (3.1)$$

Here $x \equiv (\mathbf{x}, \tau)$ comprises real-space position \mathbf{x} and imaginary time τ and $\int d\mathbf{x} \equiv \int_V d\mathbf{x} \int_0^{1/T} d\tau$. $\bar{\psi}_\sigma$ and ψ_σ are Grassmann-valued fields for electrons with spin projection σ and the first term in Eq. (3.1) describes free electrons with chemical potential μ .¹⁹ $A_\mu \equiv (\phi, -A_i)$ ($\mu = 0, 1, 2, 3$; $i = 1, 2, 3$) denotes the electromagnetic potential, with ϕ the scalar potential and $\mathbf{A} = (A_1, A_2, A_3)$ the vector potential, and the second term in Eq. (3.1) describes the free electromagnetic field in Feynman gauge [i.e., $\rho = 1$ in Eq. (A4c)].²⁰ Note that both four-vector-potential fields in the second term carry covariant indices, that is, the A action is Euclidian.²¹ $n(\mathbf{x}) = \bar{\psi}_\sigma(\mathbf{x}) \psi_\sigma(\mathbf{x})$ and $\mathbf{n}_s(\mathbf{x}) = \bar{\psi}_{\sigma_1}(\mathbf{x}) \boldsymbol{\sigma}_{\sigma_1 \sigma_2} \psi_{\sigma_2}(\mathbf{x})$ are the electronic number and spin density,¹⁴ respectively, with $\boldsymbol{\sigma}$

$=(\sigma^x, \sigma^y, \sigma^z)$ the Pauli matrices, and the third term in Eq. (3.1) describes the coupling of the electrons to the electromagnetic field, with $\mathbf{B}=\nabla\times\mathbf{A}$ the magnetic induction.²² Finally, the last term in Eq. (3.1) describes the spin-orbit interaction. Note that both terms coupling the scalar potential φ to the fermions carry an extra factor of i compared to what one might expect from the first quantized Hamiltonian. This has the same origin as the Euclidian metric mentioned above.²¹

Equation (3.1) describes a continuum model. Some of the effects we are interested in are present only in the presence of certain types of lattices and we will comment later on the modifications that occur if the electrons are put on a lattice.

B. Integrating out the photons

The action, Eq. (3.1), depends only bilinearly on the electromagnetic potential. The latter can therefore be integrated out exactly, albeit at the expense of creating four-fermion terms. The latter represent electron-electron interactions that are mediated by the exchange of virtual photons. Technically, we need the photon propagator, which we can read off the second term in Eq. (3.1),

$$\langle A_\mu(x)A_\nu(y) \rangle = \delta_{\mu\nu}\mathcal{D}(x-y) \quad (3.2a)$$

with

$$\mathcal{D}(x-y) = -4\pi(\partial_\tau^2/c^2 + \nabla^2)^{-1}\delta(x-y) \quad (3.2b)$$

or in Fourier space,

$$\mathcal{D}(k) = 4\pi/(\Omega_n^2/c^2 + \mathbf{k}^2). \quad (3.2c)$$

Here $k \equiv (i\Omega_n, \mathbf{k})$ comprises a bosonic Matsubara frequency $\Omega_n = 2\pi Tn$ and a wave vector \mathbf{k} .

The result of integrating out the photons exactly is very complicated and involves interacting electronic modes in both the spin-singlet and spin-triplet channels, the particle-particle and particle-hole channels, and all angular momentum channels. We will restrict ourselves to those terms that are most relevant for magnetism, i.e., interactions between spin-density fluctuations or modes in the s -wave particle-hole spin-triplet channel.

1. $O(1/c^0)$: Exchange interaction

We organize the various contributions to the effective electron-electron interaction in powers of $1/c$. To zeroth order only the term coupling φ to the number density n in Eq. (3.1) contributes. Integrating out φ leads to a Coulomb interaction

$$S_C = -\frac{1}{2} \int dx dy n(x) v_C(x-y) \delta(\tau_x - \tau_y) n(y) \quad (3.3a)$$

with

$$v_C(\mathbf{x}) = e^2/|\mathbf{x}|. \quad (3.3b)$$

Here we have neglected the dynamical nature of the photon propagator \mathcal{D} and have replaced it by its value at $\Omega_n=0$. The reason for this approximation is that Fermi-liquid effects lead to a dynamical screening of the Coulomb interaction that is a

much larger effect than the relativistic dynamics inherent in Eqs. (3.2).

Equation (3.3a) contains number-density fluctuations at all wavelengths. If one restricts the theory to interactions between long-wavelength fluctuations, then this interaction can be rewritten as a sum of parts that includes an interaction between spin-density fluctuations, see Ref. 23 and Appendix B. The basic point is that an interaction between number-density fluctuations at large wave numbers can be written as one between spin-density fluctuations at small wave numbers. In an effective low-energy theory that contains only fluctuations at wave numbers smaller than some cutoff λ , Eq. (3.3a) therefore contains a contribution

$$S_{\text{ex}} = \frac{\Gamma_t}{2} \int' dx \mathbf{n}_s(x) \cdot \mathbf{n}_s(x), \quad (3.4)$$

where the prime on the integral indicates that only the small-wave-number contributions (smaller than λ) to the spin density \mathbf{n}_s are to be considered in order to avoid overcounting. As has been explained in Ref. 24, it is convenient to choose the cutoff λ as a fixed fraction of the Thomas-Fermi screening wave number and the spin-triplet interaction amplitude Γ_t is a Fermi-surface average over $v_C(\mathbf{k}-\mathbf{p})\Theta(|\mathbf{k}-\mathbf{p}|-\lambda)$, with \mathbf{k} and \mathbf{p} pinned to the Fermi surface. The restriction to small wave numbers will be understood from now on and we will drop the prime on integrals.

S_{ex} is the exchange interaction between electronic spin-density fluctuations that leads ferromagnetism. For later reference we note that Γ_t is dimensionally an energy times a volume which, in this unrenormalized theory, is on the order of a Rydberg times a Fermi volume or $\Gamma_t \approx e^2/k_F^2$.

2. $O(1/c^2)$: Dipole-dipole interaction

We now turn to terms of $O(1/c^2)$. We first consider the vector potential \mathbf{A} , which couples to the spin density via the $\mathbf{B} \cdot \mathbf{n}_s$ term in Eq. (3.1). Since the coupling is directly to the spin channel, no phase-space decomposition is necessary and integrating out \mathbf{A} proceeds as in the classical case, except that the \mathbf{A} propagator now is frequency dependent, see Eq. (3.2). We will comment on the consequences of this frequency dependence in Sec. III C 2 below. Neglecting the dynamical aspects of the dipole-dipole interaction for now, we obtain a contribution to the effective action

$$S_{\text{d-d}} = 2\pi\mu_B^2 \int dx \mathbf{n}_s(x) \cdot \mathbf{n}_s(x) + \frac{\mu_B^2}{2} \int dx dy \delta(\tau_x - \tau_y) n_s^i(x) d_{ij}(\mathbf{x}-\mathbf{y}) n_s^j(y) \quad (3.5)$$

with d_{ij} from Eq. (2.7b). The first term has the same form as the exchange interaction, Eq. (3.4), but is much smaller, as $\mu_B^2 \approx \Gamma_T(v_F/c)^2$. The second one is the dipole-dipole interaction between the electron spins; if one replaces the electronic spin density by its quantum mechanical and thermal average one recovers the classical dipole-dipole term in Eq. (2.4a) or Eq. (2.7a).

3. $O(1/c^2)$: Dzyaloshinsky-Moriya and related interactions

We now return to the effects of integrating out the scalar potential φ . To $O(1/c^2)$ the relevant contribution comes from the cross term that multiplies the Coulomb (φn) term and the spin-orbit (last) term in Eq. (3.1). Contracting the two scalar potentials, integrating by parts, using Eq. (C2b), and keeping only terms that are bilinear in phase-space spin-density fluctuations, we obtain a contribution to the effective action

$$S_{\text{s-o}} = \frac{-\mu_B^2}{2} \epsilon_{ilm} \epsilon_{ijk} \int dx dy \mathcal{D}(x-y) \bar{\psi}_{\sigma_1}(x) \sigma_{\sigma_1 \sigma_4}^l \times \left[\frac{\partial}{\partial y_k} \psi_{\sigma_4}(y) \right] \left[\frac{\partial}{\partial y_j} \bar{\psi}_{\sigma_3}(y) \right] \sigma_{\sigma_3 \sigma_2}^m \psi_{\sigma_2}(x). \quad (3.6)$$

After a Fourier transform, Eq. (3.6) can be written

$$S_{\text{s-o}} = \frac{\mu_B^2}{4} \left(\frac{T}{V} \right)^2 \sum_{q,k,p} \mathcal{D}(\mathbf{q}) (\boldsymbol{\sigma}_{\sigma_1 \sigma_4} \times \boldsymbol{\sigma}_{\sigma_3 \sigma_2}) \cdot [(\mathbf{q} \times \mathbf{p}) \bar{\psi}_{\sigma_1}(k-q/2) \bar{\psi}_{\sigma_3}(p+q/2) \psi_{\sigma_4}(p-q/2) \psi_{\sigma_2}(k+q/2) - (\mathbf{q} \times \mathbf{k}) \bar{\psi}_{\sigma_1}(k+q/2) \bar{\psi}_{\sigma_3}(p-q/2) \psi_{\sigma_4}(p+q/2) \psi_{\sigma_2}(k-q/2)]. \quad (3.7)$$

Here $\mathbf{k} = (\mathbf{k}, i\omega_n)$ comprises a wave vector \mathbf{k} and a fermionic Matsubara frequency $\omega_n = 2\pi T(n+1/2)$, and \mathbf{p} and \mathbf{q} are used analogously. At this point we generalize to an effective interaction amplitude $\mathcal{D}_{\mathbf{k},\mathbf{p}}(\mathbf{q})$ that depends on \mathbf{k} and \mathbf{p} in addition to \mathbf{q} . Such a structure is generated in perturbation theory from the bare theory, where the interaction amplitude is simply given by the gauge-field propagator, as we demonstrate in Appendix D. We then have

$$S_{\text{s-o}} = \frac{\mu_B^2}{4} \left(\frac{T}{V} \right)^2 \sum_{q,k,p} [\mathcal{D}_{\mathbf{k},\mathbf{p}}(\mathbf{q})(\mathbf{q} \times \mathbf{p}) + \mathcal{D}_{\mathbf{p},\mathbf{k}}(-\mathbf{q})(\mathbf{q} \times \mathbf{k})] \cdot (\boldsymbol{\sigma}_{\sigma_1 \sigma_4} \times \boldsymbol{\sigma}_{\sigma_3 \sigma_2}) \times \bar{\psi}_{\sigma_1}(k-q/2) \bar{\psi}_{\sigma_3}(p+q/2) \psi_{\sigma_4}(p-q/2) \psi_{\sigma_2}(k+q/2). \quad (3.8)$$

Hermiticity requires $\mathcal{D}_{\mathbf{k},\mathbf{p}}^*(\mathbf{q}) = \mathcal{D}_{\mathbf{k},\mathbf{p}}(-\mathbf{q})$ [see also Eq. (B3)]. We now employ the phase-space decomposition explained in Appendix B and focus on the large-angle scattering term, Eq. (B4b). Projecting again on the spin density we obtain

$$S_{\text{s-o}} = \frac{\mu_B^2}{4} \frac{T}{V} \sum_q \mathbf{D}(\mathbf{q}) \cdot [\mathbf{n}_s(q) \times \mathbf{n}_s(-q)], \quad (3.9a)$$

where

$$\begin{aligned} \mathbf{D}(\mathbf{q}) = & \frac{T}{V} \sum_{\mathbf{k},\mathbf{p}} g_k g_p (\mathbf{k} \times \mathbf{p}) [\mathcal{D}_{(k+p+q)/2,(k+p-q)/2}(\mathbf{k}-\mathbf{p}) \\ & + \mathcal{D}_{(k+p-q)/2,(k+p+q)/2}(\mathbf{p}-\mathbf{k})] + \frac{T}{2V} \sum_{\mathbf{k},\mathbf{p}} g_k g_p (\mathbf{k}-\mathbf{p}) \\ & \times \mathbf{q} [\mathcal{D}_{(k+p+q)/2,(k+p-q)/2}(\mathbf{k}-\mathbf{p}) \\ & - \mathcal{D}_{(k+p-q)/2,(k+p+q)/2}(\mathbf{p}-\mathbf{k})] \end{aligned}$$

$$\equiv \mathbf{D}^{(1)}(\mathbf{q}) + \mathbf{D}^{(2)}(\mathbf{q}). \quad (3.9b)$$

Here

$$g_k = \epsilon_F \delta(\epsilon_k - \epsilon_F) \quad (3.10)$$

is a function that results from the projection onto the spin density and pins \mathbf{k} and \mathbf{p} to the Fermi surface.²⁵ Eq. (3.9a) has the form of the result obtained by Moriya.⁶ In what follows, we discuss the nature of the vector $\mathbf{D}(\mathbf{q})$ in some more detail.

$\mathbf{D}^{(2)}$ has the form

$$\mathbf{D}^{(2)}(\mathbf{q}) = i\mathbf{q} \times \mathbf{d}^{(2)} + O(\mathbf{q}^3) \quad (3.11a)$$

with $\mathbf{d}^{(2)}$ a real vector given by

$$\begin{aligned} \mathbf{d}^{(2)} = & \frac{T}{2iV} \sum_{\mathbf{k},\mathbf{p}} g_k g_p (\mathbf{p}-\mathbf{k}) [\mathcal{D}_{(k+p)/2,(k+p)/2}(\mathbf{k}-\mathbf{p}) \\ & - \mathcal{D}_{(k+p)/2,(k+p)/2}(\mathbf{p}-\mathbf{k})]. \end{aligned} \quad (3.11b)$$

$\mathbf{D}^{(1)}$ can be written

$$D_i^{(1)}(\mathbf{q}) = i D_{ij} q_j + O(\mathbf{q}^3) \quad (3.12a)$$

with D_{ij} a real rank-2 tensor given by

$$D_{ij} = \sum_{\mathbf{k},\mathbf{p}} g_k g_p \epsilon_{ilm} k_l p_m \mathcal{E}_j(\mathbf{k},\mathbf{p}), \quad (3.12b)$$

where

$$\begin{aligned} \mathcal{E}_i(\mathbf{k},\mathbf{p}) = & \frac{-1}{2} \int dx dy (x_i - y_i) e^{-i(\mathbf{k}+\mathbf{p})(\mathbf{x}+\mathbf{y})/2} \\ & \times \int dz \sin[(\mathbf{k}-\mathbf{p}) \cdot \mathbf{z}] \mathcal{D}(\mathbf{x},\mathbf{y};\mathbf{z}) \end{aligned} \quad (3.12c)$$

with $\mathcal{D}(\mathbf{x},\mathbf{y};\mathbf{z})$ the Fourier transform of $\mathcal{D}_{\mathbf{k},\mathbf{p}}(\mathbf{q})$ in analogy to Eq. (B2b). D_{ij} has a symmetric part and an antisymmetric part. The latter can be combined with $\mathbf{D}^{(2)}$ above to form a contribution to \mathbf{D} that we denote by

$$\mathbf{D}^{(-)}(\mathbf{q}) = i\mathbf{q}\mathbf{d} + O(\mathbf{q}^3), \quad (3.13a)$$

where

$$d_i = d_i^{(2)} + \frac{T}{2iV} \sum_{\mathbf{k},\mathbf{p}} g_k g_p (p_i k_j - k_i p_j) \mathcal{E}_j(\mathbf{k},\mathbf{p}). \quad (3.13b)$$

The symmetric part can be written as a diagonal tensor plus a traceless rank-2 tensor and the latter can be diagonalized by means of an orthogonal transformation that amounts to a spatial rotation. The symmetric part of D_{ij} we thus can write

$$D_{ij}^{(+)} = \frac{1}{3} \text{tr} D \delta_{ij} + a_i \delta_{ij} \quad (3.14a)$$

(no summation convention) where the a_i obey

$$\sum_i a_i = 0. \quad (3.14b)$$

If desirable, the a_i can be explicitly constructed from Eq. (3.12b). We thus have a second contribution to \mathbf{D} that we denote by

$$\mathbf{D}^{(+)}(\mathbf{q}) = \frac{i}{3} \text{tr } D\mathbf{q} + i \begin{pmatrix} a_x q_x \\ a_y q_y \\ a_z q_z \end{pmatrix} + O(\mathbf{q}^3) \quad (3.15)$$

and

$$\mathbf{D}(\mathbf{q}) = \mathbf{D}^{(+)}(\mathbf{q}) + \mathbf{D}^{(-)}(\mathbf{q}). \quad (3.16)$$

Combining our results and transforming back to real space, we now have

$$S_{s-o} = S_{DM} + S'_{DM} + S''_{DM}, \quad (3.17a)$$

where

$$S_{DM} = \frac{\mu_B^2}{12} \text{tr } D \int dx \mathbf{n}_s(x) \cdot [\nabla \times \mathbf{n}_s(x)], \quad (3.17b)$$

$$S'_{DM} = \frac{\mu_B^2}{4} \int dx \mathbf{n}_s(x) \cdot \begin{pmatrix} a_x \partial_x \\ a_y \partial_y \\ a_z \partial_z \end{pmatrix} \mathbf{n}_s(x), \quad (3.17c)$$

$$S''_{DM} = \frac{\mu_B^2}{4} \int dx [\nabla \cdot \mathbf{n}_s(x)] [\mathbf{d} \cdot \mathbf{n}_s(x)]. \quad (3.17d)$$

S_{DM} is the Dzyaloshinsky-Moriya interaction that is believed to be responsible for the helimagnetism observed in MnSi and FeGe.⁸ S'_{DM} is a closely related term but absent in systems with cubic lattices (e.g., MnSi or FeGe) due to the constraint, Eq. (3.14b). Finally, S''_{DM} is another term that is allowed by symmetry and is generated by the above derivation. All three of these terms are contained in Moriya's general result,⁶ which takes the form of our Eq. (3.9a) above but the effects of S'_{DM} and S''_{DM} have, to our knowledge, not been discussed explicitly.

Note that a necessary condition for any of these interactions to be nonzero is that the system is not invariant under parity: both $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ can be nonzero only if $\mathcal{D}_{k,p}(\mathbf{q})$ is odd under $\mathbf{q} \rightarrow -\mathbf{q}$ or, equivalently, if $\mathcal{D}(\mathbf{x}, \mathbf{y}; \mathbf{z})$ is odd under $\mathbf{z} \rightarrow -\mathbf{z}$. This implies that the DM interaction requires a lattice that is not invariant under spatial inversion; in any continuum model, where the electron-electron interaction is necessarily even under parity, it vanishes. See Appendix D and Sec. IV for further discussions of this point.

C. Fermionic action and magnetic order parameter

We now have the following result. After integrating out the photons, the effective fermionic action reads

$$\begin{aligned} S_{\text{eff}}[\bar{\psi}, \psi, \mathbf{M}] &= S_0 + S'_{\text{int}} + S_{\text{ex}} + S_{\text{d-d}} + S_{DM} + S'_{DM} + S''_{DM} \\ &\equiv S'_0 + S_{\text{int}}^t. \end{aligned} \quad (3.18)$$

Here S_0 describes noninteracting electrons (either free electrons or band electrons, depending on the model considered) and S'_{int} contains all interactions between modes other than the spin density, which we have not explicitly considered with the exception of the Coulomb interaction, Eq. (3.3). Collectively we denote these two terms by S'_0 , S_{ex} , $S_{\text{d-d}}$, S_{DM} ,

, and S''_{DM} are the exchange, dipole-dipole, and Dzyaloshinsky-Moriya interactions given by Eqs. (3.4) and (3.5), respectively. S_{ex} and $S_{\text{d-d}}$ are always present, which, if any, of the terms in S_{DM} are nonzero depends on the details of the lattice structure and absence of spatial inversion symmetry is a prerequisite for any of them to be nonzero. Collectively, we denote the sum of these interactions in the spin density or triplet channel by S_{int}^t .

1. Structure of a magnetic order-parameter description

For applications such as fermionic cold gases one will want to work directly with the fermionic action. For applications to magnets it is convenient to introduce a composite field $\mathbf{M}(x)$ whose expectation value is proportional to the magnetization. To this end we write

$$S_{\text{int}}^t = \frac{1}{2} \int dx dy n_s^i(x) \Gamma_{ij}(x-y) n_s^j(y) \quad (3.19a)$$

with

$$\begin{aligned} \Gamma_{ij}(x-y) &= \delta(x-y) \Gamma_t + \mu_B^2 \delta(\tau_x - \tau_y) d_{ij}(x-y) + \frac{\mu_B^2}{2} \delta(x-y) \\ &\times \left[\frac{1}{3} \text{tr } D \epsilon_{ikj} \partial_k + \frac{1}{2} \epsilon_{ikj} (a \partial)_k + \frac{1}{2} d_i \partial_j \right]. \end{aligned} \quad (3.19b)$$

Here $(a \partial)_k = a_k \partial_k$ (no summation convention). We now decouple Eq. (3.19a) by means of a Hubbard-Stratonovich transformation with a bosonic field $\mathbf{M}(x)$. Neglecting a constant contribution to the action, this allows us to write

$$\begin{aligned} S_{\text{eff}}[\bar{\psi}, \psi, \mathbf{M}] &= S'_0[\bar{\psi}, \psi] - \frac{1}{2} \int dx dy M_i(x) \Gamma_{ij}(x-y) M_j(y) \\ &+ \frac{1}{4} \int dx dy [M_i(x) \Gamma_{ij}(x-y) n_s^j(y) + n_s^i(x) \Gamma_{ij} \\ &\times (x-y) M_j(y)]. \end{aligned} \quad (3.20)$$

If one neglects the interacting part of S'_0 this action depends only bilinearly on the fermion fields, and one can formally integrate out the fermions in order to obtain a theory entirely in terms of the order-parameter field \mathbf{M} . This is a generalization of, and replaces, the Hertz-Millis theory.^{10,11} However, in general this is not a good strategy since it amounts to integrating out soft excitations, which means that any order-parameter theory will in general not be well behaved. Physically, these soft quasiparticle excitations can change the nature of the phase transition¹² or they themselves can become critical.²⁶ In either case they must be treated on equal footing with the order-parameter fluctuations. It therefore is technically advantageous, and physically more transparent, to work with the coupled field theory represented by Eq. (3.20).

2. Comments on the magnetization dynamics

We conclude this section with a brief discussion of the dynamics of the dipole-dipole interaction, which we neglected in Sec. III B 2. If we restore the frequency depen-

dence of the photon propagator and expand in powers of the frequency, then the leading dynamical contribution of the dipole-dipole interaction to Eq. (3.20) takes the form

$$S_{\text{d-d}}^{\text{dyn}} = \mu_B^2 \frac{T}{V} \sum_k (\delta_{ij} - \hat{k}_i \hat{k}_j) \frac{\Omega_n^2}{c^2 k^2} M_i(k) M_j(-k). \quad (3.21)$$

As long as Ω_n scales as $|\mathbf{k}|$, this scales the same as the $|\Omega_n|/|\mathbf{k}|$ term in the order-parameter theory that is induced by Fermi-liquid effects¹⁰ but has a prefactor that is smaller by a factor of $(v_F/c)^2$. However, in classical dipolar magnets the order parameter is known to no longer be conserved.⁴ That is, Ω scales as a constant and this should be reflected in the quantum theory as well, although it is currently not known how this is realized. This suggests that the contribution shown in Eq. (3.21) dominates the Fermi-liquid-induced dynamics in a scaling sense, although it has a small prefactor and will become important at sufficiently long time scales.

IV. DISCUSSION AND CONCLUSION

We now discuss the significance of various interactions for a number of problems.

A. Energy scales and the significance of the dipole-dipole interaction

As we mentioned in Sec. III A, the energy scale for the exchange interaction in the bare theory is the atomic scale or roughly 100 000 K. The corresponding length scale is on the order of 1 Å. This is not consistent with the experimental fact that magnetic ordering is observed only at much lower temperatures; e.g., on the order of 1000 K in Fe and Ni, on the order of room temperature in FeGe, and on the order of 30 K in MnSi. The reason for this discrepancy lies in the fact that the bare theory is renormalized in quantitatively substantial ways, and the corresponding energy and length scales in the properly renormalized theory are consistent with experimental observations.²

Equations (3.5) show that the dipole-dipole and DM interactions, within the framework of the bare theory, are weaker than the exchange interaction by a factor of $(v_F/c)^2$ or about 10^{-4} .²⁷ Relative to the bare exchange interaction, this implies an energy scale on the order of 10 K, which is comparable with the ordering temperature in MnSi. On the other hand, the length scale associated with the DM interaction (i.e., the pitch length of the spin helix²⁸) is only about 200 Å (in MnSi) (Ref. 29) to 700 Å (in FeGe) (Ref. 30) or only a factor of 10^2 – 10^3 larger than the atomic length scale.

These observations indicate that there are strong renormalizations, due to band-structure and many-body effects, of all terms in the bare action and that different terms are renormalized in different ways in different materials. While this makes it hard to make general statements, the bare theory suggests that the DM interactions and the dipole-dipole interaction are generically comparable in strength and in MnSi, for instance, both are expected to be a substantial fraction of the (greatly reduced by renormalizations) exchange interaction. We thus conclude that there is no *a priori* reason to neglect the dipole-dipole interaction in any system where the

DM interaction is known to be important. This calls for a re-evaluation of a number of interesting problems, some of which we list in the following section.

B. Significance of the dipole-dipole interaction

We conclude by discussing a number of problems where the dipole-dipole interaction is either known to be important, or might be important, with an emphasis on low-temperature magnets and other fermion systems.

(1) Classical Heisenberg ferromagnets. This problem was worked on in great detail by Aharony and Fisher⁵ for the static critical behavior and by Frey and Schwab¹⁴ for the dynamical critical behavior. The renormalization group done by Aharony and Fisher started from a nonlocal order-parameter theory, Eq. (2.7), which leads to a somewhat non-conventional renormalization procedure. The nonlocality in Eq. (2.7) is due to integrating out the gauge-field fluctuations or photons. It would be interesting to repeat this calculation starting from the coupled local-field theory given by our Eq. (2.2) before these fluctuations are integrated out.

(2) Classical helimagnets. The standard phase-transition treatment for helimagnetism due to the DM interaction is due to Bak and Jensen.⁸ Neglecting the dipole-dipole interaction term they conclude that there is a fluctuation-induced first-order phase transition from paramagnetism to helimagnetism. An interesting question is whether or not the dipole-dipole interaction modifies this conclusion. This seems especially relevant for MnSi where the phase transition is at low temperatures, i.e., it is a weak helimagnet.

(3) It has been shown that in clean itinerant ferromagnets, the ferromagnetic transition is generically of first order at zero temperature.^{12,31} This conclusion ignores the effects of the dipole-dipole interaction terms. It would be very interesting to investigate if dipolar interactions can modify this generic conclusion.

(4) Phase ordering is an important problem in ferromagnets.³² The dipole-dipole interaction terms has not been included in either the classical or quantum (zero-temperature) ferromagnetic phase ordering problems. Simple arguments indicate it will be important.

(5) Fermionic cold atom systems. Recently there has been a considerable amount of work on gases of fermions with dipolar interactions, see Ref. 13 and references therein. These systems are important for fermions in optical lattices. The dipolar interactions also serve as a mechanism for liquid crystal like phase formation in fermion systems.

(6) The dipole-dipole interaction term is important in the dynamics of classical antiferromagnets, both in the ordered phase, and near or at the phase transition if the systems is below its upper critical dimension.⁴ Simple considerations suggest that they will also be important in low-dimensional (1+1 or 2+1) itinerant quantum antiferromagnets; see Sec. III C 2 above for one aspect of this problem.

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APPENDIX A: THE COMPLETE ACTION TO $O(1/c^2)$

In this appendix we give the complete action for electrons interacting with electromagnetic fields in the weakly relativistic limit, up to and including terms of $O(1/c^2)$. Let $A_\mu = (\varphi, -\mathbf{A})$ be the four-vector potential, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ the Pauli matrices, and $\mu_B = e/2m_e c$ the Bohr magneton. We use standard relativistic notation, with covariant and contravariant indices related by a Minkowski metric $g_{\mu\nu} = (+, -, -, -)$. Expanding the Dirac equation in powers of $1/c$, one obtains, to order $1/c^2$, the following Hamiltonian in first quantization:³³⁻³⁵

$$\hat{H} = \hat{H}_P + \hat{H}_{so} + \hat{H}_D + \hat{H}_{\delta m}. \quad (\text{A1a})$$

Here \hat{H}_P is the Pauli Hamiltonian,

$$\hat{H}_P = \frac{1}{2m_e} \left[-i \nabla - \frac{e}{c} \mathbf{A}(\mathbf{x}, t) \right]^2 + e\varphi(\mathbf{x}, t) - \mu_B \boldsymbol{\sigma} \cdot [\nabla \times \mathbf{A}(\mathbf{x}, t)] \quad (\text{A1b})$$

and

$$\hat{H}_{so} = \frac{ie}{4m_e^2 c^2} \boldsymbol{\sigma} \cdot [\nabla \varphi(\mathbf{x}, t) \times \nabla], \quad (\text{A1c})$$

$$\hat{H}_D = \frac{-e}{8m_e^2 c^2} \nabla^2 \varphi(\mathbf{x}, t), \quad (\text{A1d})$$

$$\hat{H}_{\delta m} = \frac{-1}{8m_e^2 c^2} \nabla^4 \quad (\text{A1e})$$

describe the spin-orbit interaction, the Darwin term, and the relativistic mass correction, respectively. Via standard techniques,^{19,20} this theory can be reformulated in terms of an action that depends on fermionic (i.e., Grassmann-valued) field ψ and its adjoint $\bar{\psi}$ as well as the four-vector-potential field A_μ . For the partition function one obtains

$$Z = Z_A Z_{FP}, \quad (\text{A2})$$

where

$$Z_A = \int D[\bar{\psi}, \psi] D[A_\mu] e^{S_A[\bar{\psi}, \psi; A_\mu]} \quad (\text{A3})$$

with an action

$$S_A[\bar{\psi}, \psi; A_\mu] = S_\psi[\bar{\psi}, \psi] + S_A[A_\mu] + S_c[\bar{\psi}, \psi; A_\mu]. \quad (\text{A4a})$$

Here

$$S_\psi[\bar{\psi}, \psi] = \int dx \bar{\psi}_\sigma(x) \left[-\partial_\tau + \frac{1}{2m_e} \nabla^2 + \mu - \frac{1}{8m_e^2 c^2} \nabla^4 \right] \psi_\sigma(x) \quad (\text{A4b})$$

describes the electrons with chemical potential μ and

$$S_A[A_\mu] = \frac{1}{8\pi} \int dx A_\mu(x) \left[\frac{1}{c^2} \partial_\tau^2 + \nabla^2 \right] A_\mu(x) + \frac{1}{8\pi} \frac{\rho-1}{\rho} \int dx \left[\frac{1}{c} \partial_\tau + \nabla \cdot \mathbf{A}(x) \right]^2 \quad (\text{A4c})$$

describes the electromagnetic fields, with $\rho \in \mathcal{R}$ a gauge fixing parameter. We use a four-vector notation $x \equiv (\tau, \mathbf{x})$, $\int dx \equiv \int d\tau \int d\mathbf{x}$ for space and imaginary time. S_c describes the coupling between the fermions and the electromagnetic field; it contains four separate contributions,

$$S_c[\bar{\psi}, \psi; A_\mu] = S_{c,P} + S_{c,L} + S_D + S_{so}. \quad (\text{A4d})$$

Here

$$S_{c,P}[\bar{\psi}, \psi; A_\mu] = -ie \int dx \varphi(x) n(x) + \mu_B \int dx \mathbf{B}(x) \cdot \mathbf{n}_s(x) \quad (\text{A4e})$$

is the Coulomb and Zeeman paramagnetic coupling that is included in the Pauli equation, with $n(x) = \bar{\psi}_\sigma(x) \psi_\sigma(x)$ and $\mathbf{n}_s(x) = \bar{\psi}_{\sigma_1}(x) \boldsymbol{\sigma}_{\sigma_1 \sigma_2} \psi_{\sigma_2}(x)$.

$$S_{c,L}[\bar{\psi}, \psi; A_\mu] = -i2\mu_B \int dx \bar{\psi}_\sigma(x) \mathbf{A}(x) \cdot \nabla \psi_\sigma(x) + i\mu_B \int dx [\nabla \cdot \mathbf{A}(x)] n(x) - \frac{e^2}{2m_e c^2} \int dx A^2(x) n(x) \quad (\text{A4f})$$

is the Landau diamagnetic coupling;

$$S_D[\bar{\psi}, \psi; A_\mu] = \frac{-ie}{8m_e^2 c^2} \int dx [\nabla^2 \varphi(x)] n(x) \quad (\text{A4g})$$

is the Darwin term that, in the relativistic hydrogen atom, leads to the so-called *zitterbewegung*, and

$$S_{so}[\bar{\psi}, \psi; A_\mu] = \frac{-e}{4m_e^2 c^2} \int dx \bar{\psi}_{\sigma_1}(x) \boldsymbol{\sigma}_{\sigma_1 \sigma_2} \cdot [\nabla \varphi(x) \times \nabla] \psi_{\sigma_2}(x) \quad (\text{A4h})$$

is the spin-orbit coupling.

The second factor in Eq. (A2) is

$$Z_{FP} = \int D[\bar{\eta}, \eta] e^{-S_{FP}[\bar{\eta}, \eta]}. \quad (\text{A5a})$$

Here η is a one-component Grassmannian field known as a Fadeev-Popov ghost field, with $\bar{\eta}$ its adjoint, which is governed by an action

$$S_{FP}[\bar{\eta}, \eta] = \int dx \bar{\eta}(x) \partial_\mu \partial^\mu \eta(x). \quad (\text{A5b})$$

APPENDIX B: PHASE-SPACE DECOMPOSITION OF INTERACTION TERMS

For completeness, in this appendix we briefly recapitulate the arguments that lead to the generation of a spin-spin in-

teraction, Eq. (3.4), from a density-density interaction, Eq. (3.3). For further discussion, see Refs. 23, 24, and 36; the latter also explain the relation to the work by Shankar.³⁷

Consider an electron-electron interaction with an interaction amplitude W . For simplicity we assume that the interaction is purely static, and translationally invariant, but otherwise general. The action has the form

$$S = \frac{-1}{2} \int dx_1 \cdots dx_4 \int d\tau W(x_1 - x_2, x_3 - x_4; (x_3 + x_4 - x_1 - x_2)/2) \tau_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \bar{\psi}_{\sigma_1}(x_1, \tau) \bar{\psi}_{\sigma_3}(x_3, \tau) \times \psi_{\sigma_4}(x_4, \tau) \psi_{\sigma_2}(x_2, \tau) \quad (\text{B1})$$

with τ a general rank-4 tensor. We define Fourier transforms

$$\bar{\psi}_{\sigma}(k) = \sqrt{T/V} \sum_k e^{-ikx} \bar{\psi}_{\sigma}(k), \quad \psi_{\sigma}(k) = \sqrt{T/V} \sum_k e^{ikx} \psi_{\sigma}(k), \quad (\text{B2a})$$

$$W_{k,p}(q) = \int dx dy dz W(x, y, z) \quad (\text{B2b})$$

with $kx = \mathbf{k} \cdot \mathbf{x} - \omega_n \tau$ where \mathbf{k} is a wave vector and $\omega_n = 2\pi T(n+1/2)$ a fermionic Matsubara frequency. Hermiticity requires

$$W_{k,p}^*(q) = W_{k,p}(-q), \quad (\text{B3a})$$

$$\tau_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}^* = \tau_{\sigma_2 \sigma_1 \sigma_4 \sigma_3}. \quad (\text{B3b})$$

We then have

$$S = \frac{-1}{2} \left(\frac{T}{V} \right)^2 \sum_{k,p,q} W_{k,p}(q) \tau_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \bar{\psi}_{\sigma_1}(k - q/2) \bar{\psi}_{\sigma_3}(p + q/2) \psi_{\sigma_2}(p - q/2) \psi_{\sigma_2}(k + q/2) \quad (\text{B4a})$$

$$= \frac{-1}{2} \left(\frac{T}{V} \right)^2 \sum_{k,p,q} W_{(k+p-q)/2, (k+p+q)/2}(p - k) \tau_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \bar{\psi}_{\sigma_1}(k - q/2) \bar{\psi}_{\sigma_3}(p + q/2) \psi_{\sigma_4}(k + q/2) \psi_{\sigma_2}(p - q/2) \quad (\text{B4b})$$

$$= \frac{-1}{2} \left(\frac{T}{V} \right)^2 \sum_{k,p,q} W_{(p-k+q)/2, (k-p+q)/2}(p + k) \tau_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \bar{\psi}_{\sigma_1}(-k + q/2) \bar{\psi}_{\sigma_3}(k + q/2) \times \psi_{\sigma_4}(-p + q/2) \psi_{\sigma_2}(p + q/2). \quad (\text{B4c})$$

As long as all wave vectors are summed over, all three of these expressions are identical. If one restricts the summation in such a way that both $|q|$ and the modulus of the third argument of W are smaller than a cutoff wave number λ , then we can represent the action as a sum of all three terms. They represent small-angle scattering, large-angle scattering, and $2k_F$ scattering, respectively. Alternatively, if one is interested in only one of these channels, one can pick the appropriate formulation of S , restrict oneself to small wave numbers, and neglect the other channels. For our purposes, we are interested in the large-angle scattering channel, Eq.

(B4b). By choosing $\tau_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} = \sigma_{\sigma_1 \sigma_2}^0 \sigma_{\sigma_3 \sigma_4}^0$ and making use of Eq. (C2a) we obtain a term that has the structure of the exchange interaction, Eq. (3.4) (in addition to a contribution to the number-density interaction). The resulting spin-triplet mode is more complicated than a pure spin density but it has an overlap with the spin density and can be restricted to the latter by the projection technique explained in Ref. 36. Note that a repulsive Coulomb interaction results in an attractive exchange interaction due to a commutation of fermion fields that is necessary to write the result in the form of Eq. (3.4). Similarly, by choosing $\tau_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} = \sigma_{\sigma_1 \sigma_2}^0 \boldsymbol{\sigma}_{\sigma_3 \sigma_4}$ and making use of Eq. (C2b) we obtain the structure found in Sec. III B 3 (in addition to terms that couple the number density and the spin density).

APPENDIX C: PROPERTIES OF PAULI MATRICES

Here we give some properties of the Pauli matrices that were used in Sec. III. Let $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z) \equiv (\sigma^1, \sigma^2, \sigma^3)$ be the Pauli matrices with the commutator property

$$\epsilon_{ijk} \sigma^i \sigma^j = i \sigma^k \quad (i, j, k = 1, 2, 3), \quad (\text{C1})$$

and σ^0 the 2×2 unit matrix. Then the following identities hold

$$\sigma_{\sigma_1 \sigma_2}^0 \sigma_{\sigma_3 \sigma_4}^0 = \frac{1}{2} \sigma_{\sigma_1 \sigma_4}^0 \sigma_{\sigma_3 \sigma_2}^0 + \frac{1}{2} \boldsymbol{\sigma}_{\sigma_1 \sigma_4} \cdot \boldsymbol{\sigma}_{\sigma_3 \sigma_2}, \quad (\text{C2a})$$

$$\sigma_{\sigma_1 \sigma_2}^0 \boldsymbol{\sigma}_{\sigma_3 \sigma_4} = \frac{1}{2} \boldsymbol{\sigma}_{\sigma_1 \sigma_4} \sigma_{\sigma_3 \sigma_2}^0 + \frac{1}{2} \sigma_{\sigma_1 \sigma_4}^0 \boldsymbol{\sigma}_{\sigma_3 \sigma_2} + \frac{i}{2} \boldsymbol{\sigma}_{\sigma_1 \sigma_4} \times \boldsymbol{\sigma}_{\sigma_3 \sigma_2}. \quad (\text{C2b})$$

Equation (C2a) is easily checked by a direct calculation and Eq. (C2b) follows by multiplying Eq. (C2a) by $\boldsymbol{\sigma}$ from the right and using Eq. (C1).

APPENDIX D: NONLOCAL ELECTRON-ELECTRON INTERACTION

In this appendix we demonstrate that a general interaction amplitude of the type used in Eq. (B1) is generated in perturbation theory from an ordinary two-body interaction. For simplicity, we consider spinless fermions interacting via an interaction potential V ; spin dependence or gradients, as in the last term of Eq. (3.1), are easily added. We also consider a translationally invariant model; we will comment on this feature below.

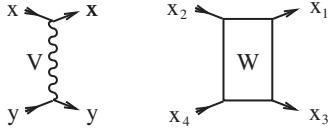
The bare electron-electron interaction is described by a term in the action

$$S = \int dx dy V(x - y) \bar{\psi}(x) \bar{\psi}(y) \psi(y) \psi(x). \quad (\text{D1})$$

Our goal is to construct an effective interaction of the form

$$S_{\text{eff}} = \int dx_1 \cdots dx_4 \times W[x_1 - x_2, x_3 - x_4; (x_3 + x_4 - x_1 - x_2)/2] \times \bar{\psi}(x_1) \bar{\psi}(x_3) \psi(x_4) \psi(x_2) \quad (\text{D2})$$

that has a structure necessary for contributing to the tensor

FIG. 2. Bare interaction V and effective interaction W .

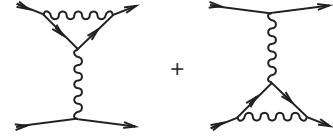
D_{ij} , Eqs. (3.12b) and (3.12c), see Fig. 2. The bare interaction corresponds to

$$W^{(1)}(x, y, z) = \delta(x) \delta(y) V(z). \quad (\text{D3})$$

This does not contribute to D_{ij} since it enforces $x=y=0$. More generally, contributions to W with the property $W(x, y, z) = W(y, x, z)$ do not contribute to D_{ij} . At second order in V , there are several diagrams, both tree-level and one-loop diagrams, that have this property. Consider, however, the diagrams shown in Fig. 3. They correspond to

$$\begin{aligned} W^{(2)}(x, y, z) = & \delta(x) V(-y) \int dx' G(-x' + y/2) G(x' + y/2) \\ & \times V(-x' + z/2) + \delta(y) V(-x) \int dx' \\ & \times G(-x' + x/2) G(x' + x/2) V(-x' + z/2), \end{aligned} \quad (\text{D4})$$

where $G(x-y) = \langle \psi(x) \bar{\psi}(y) \rangle$ is the electron propagator.

FIG. 3. A second-order contribution to W .

In order for W to contribute to D_{ij} we must have $W(x, y, -z) = -W(x, y, z)$, see Eq. (3.12c). From Eq. (D4) we see that this is the case if and only if $V(x)$ has an odd component $V^-(x) = -V(x)$. As far as the contribution to D_{ij} is concerned we can thus replace V in Eq. (D4) by V^- and this automatically ensures $W(y, x, z) = -W(x, y, z)$. If V is the propagator of a scalar field, such as the quantity \mathcal{D} in Sec. III, then these symmetry properties can obviously not be realized. However, in a realistic solid-state model V represents the screened Coulomb interaction, $V(x, y) = v_C(x-y)/\epsilon(x, y)$, and the dielectric function ϵ has a contribution from the lattice in addition to an electronic contribution. The latter will only have the symmetry of the space group of the lattice and on a lattice without inversion symmetry one will have $\epsilon(y, x) \neq \epsilon(x, y)$. Translational symmetry will also be broken, of course, but this is not necessary for making the DM interaction nonzero, as the above example shows. Note that one can consider a coarse-grained continuum theory for the DM interaction, see Eq. (2.8), but the relevant Landau coefficient *must* depend on the underlying lattice and vanish if the true continuum limit is taken.

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- ¹⁴We define the classical magnetization \mathbf{M} as the (quantum mechanical and thermal) average $\langle \bar{\psi} \mathbf{a} \psi \rangle$ (see Sec. III), so a g factor $g=2$ is already included in the definition of \mathbf{M} .
- ¹⁵Note that the results for all physical quantities are independent of the value of ρ . The characteristic anisotropic structure of the dipole tensor, Eq. (2.4b), in the effective action results from the curl in the coupling, Eq. (2.2b), and *not* from the structure of the \mathbf{A} propagator in any particular gauge. This becomes obvious if one considers that in Feynman gauge the \mathbf{A} propagator is isotropic and given by $\langle A_i(\mathbf{k}) A_j(\mathbf{k}) \rangle = \delta_{ij} 4\pi/k^2$.

¹⁶L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1985).

¹⁷Equation (2.3), in contrast to Eq. (2.2c), is valid *only* for $\rho \rightarrow 0$. This is because the constraint in Eq. (2.2c) describes the absence of magnetic monopoles and the physics changes for nonzero values of ρ .

¹⁸For certain electron systems in reduced dimensionalities other terms can appear in the first-quantized Hamiltonian, such as the Rashba spin-orbit interaction, see Refs. 38 and 39. These can be treated the same way as the terms in Eq. (3.1) to determine their effect on the effective-field theory.

¹⁹J. W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, New York, 1988).

²⁰J. I. Kaputza, *Finite-Temperature Field Theory* (Cambridge University Press, Cambridge, 1989).

²¹This change from a Minkowski metric in a zero-temperature formalism to a Euclidian one at finite temperature is the result of a Wick rotation that multiplies the timelike components of *all* four vectors by $i = \sqrt{-1}$, not just time itself. See 20 for details of this procedure.

²²The scalar potential couples to the ionic density in addition to the electronic one. This coupling, which leads to important qualitative effects in lattice systems, is neglected in the explicit development of the theory in this section but will be taken into later, see Sec. III B 3 and Appendix D.

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of *Quantum Field Theory in Statistical Physics* (Dover, New York, 1963), Sec. 18.

²⁴D. Belitz, F. Evers, and T. R. Kirkpatrick, *Phys. Rev. B* **58**, 9710 (1998).

²⁵ g_k also depends on q but its leading q dependence is quadratic and thus contributes only to terms in the action that are cubic in the gradient operator.

²⁶S. Sachdev, [arXiv:0901.4103](https://arxiv.org/abs/0901.4103) (unpublished).

²⁷Alternatively, one can write this suppression factor in terms of the fine-structure constant α as $(\alpha\epsilon_F/e^2k_F)^2 \approx \alpha^2 \approx 10^{-4}$ for typical metals, see Ref. 1.

²⁸The helical pitch wave number q is obtained by comparing the DM term, Eq. (3.17b), with the gradient-squared term in a Heisenberg model, or $\mu_B^2 q/k_F \approx \Gamma_t q^2/k_F^2$ in the bare theory, which leads to $q/k_F \approx (v_F/c)^2$. The ratio of the microscopic length scale over the DM length scale is thus on the same order as the ratio of the DM energy scale over the microscopic energy scale (not, for instance, as the square root of the energy scale ratio).

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³³In the usual expansion of the Dirac equation, some additional term naturally arise and are often included in the spin-orbit interaction. Here we keep only terms that are of $O(1/c^2)$.

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